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Some exact solutions of all $f(R_{\mu\nu})$ theories in three dimensions

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We find constant scalar curvature Type-N and Type-D solutions in all higher curvature gravity theories with actions of the form $f(R_{\mu\nu})$ that are built on the Ricci tensor, but not on its derivatives. In our construction, these higher derivative theories inherit some of the previously studied solutions of the cosmological topologically massive gravity and the new massive gravity field equations, once the parameters of the theories are adjusted. Besides the generic higher curvature theory, we have considered in some detail the examples of the quadratic curvature theory, the cubic curvature theory, and the Born-Infeld extension of the new massive gravity.

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I. INTRODUCTION

In $2 + 1$ dimensions, various higher curvature modifications of Einstein's theory, such as the new massive gravity (NMG) [1,2], a specific cubic curvature gravity [3], and the Born-Infeld gravity [4], attracted attention recently. NMG provides a unitary nonlinear extension of the Fierz-Pauli mass in both flat and maximally symmetric constant curvature backgrounds. For anti–de Sitter (AdS) backgrounds, NMG has a drawback: bulk and boundary unitarity is in conflict and hence does not fit well into the AdS/CFT picture. In [3], a cubic extension of NMG was given which again has this conflict. A simple, in principle infinite order (in curvature) extension of NMG in terms of a Born-Infeld gravity, dubbed as BINMG, was introduced in [4] which again has this bulk-boundary unitarity conflict. Finally, in [5], all bulk and boundary unitary theories in three dimensions were constructed. These theories should be at least cubic in curvature, if the contractions of Ricci tensor are used. Linearized excitations in these models have been studied.

This work is devoted to the study of some exact solutions of all $f(R_{\mu\nu})$ theories in three dimensions that include the quadratic, cubic, and BINMG theories as subclasses. For the quadratic and cubic curvature theories, we will not restrict ourselves to the unitary models but study the most generic theories. Some exact solutions of NMG were given in [2,6–12] (and also see [13] for the solutions in the "generalized NMG" that includes the gravitational Chern-Simons term [14,15] in addition to the Einstein and the quadratic terms). Save for the maximally symmetric solutions and AdS waves [16] and black holes, to the best of our knowledge, a general approach to the solutions of the general quadratic theory or the more general $f(R_{\mu\nu})$ theories has not appeared yet (some Type-III and Type-N solutions of the D-dimensional quadratic gravity were given in [17]). For cubic curvature theories and for BINMG some solutions were found before[1] [3,19–22]. In this paper we will give a systematic way of finding solutions in these theories. As it will be clear from the field equations of these theories, without some symmetry assumption, finding solutions is almost hopeless. The assumption of the existence of a Killing vector highly restricts the geometry in three dimensions [9], the solutions of NMG and topologically massive gravity (TMG) under this assumption include some classes of Type-N and Type-D solutions. However, even without a symmetry assumption, in addition to the above solutions, some new Type-N and Type-D solutions of NMG were found in [10–12] using the tetrad formalism. Furthermore, in [10–12], the solutions of NMG inherited from the solutions of TMG are found by relating the field equations of the two theories. A similar technique will be applied in this paper to find large classes of solutions of all $f(R_{\mu\nu})$ gravity theories.

The main result of this work is to introduce a technique for finding exact solutions of any higher curvature gravity theory in three dimensions from the solutions of TMG and NMG. The technique is based on the observation that in $D = 3$ for constant scalar invariant (CSI) spacetimes[2] of Type N and Type D, the field equations of any higher curvature gravity theory with a generic Lagrangian $f(R_{\mu\nu})$ reduce to the field equations of NMG whose form is also the same as the TMG field equations written in the quadratic form. With this fact, one can obtain new solutions of $f(R_{\mu\nu})$ theories by just relating their parameters to the parameters of TMG and/or NMG. We have used the results obtained for the $f(R_{\mu\nu})$ theory for generic quadratic and cubic curvature gravity theories in addition to BINMG, and found new solutions for these theories which are

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[1]More recently, Type-N solutions of BINMG and extended NMG theories appeared in [18]

[2]The scalar invariants that we mention are the ones constructed by contractions of the Ricci tensor but not its derivatives.

inherited from the Type-N and Type-D solutions of TMG, compiled in [23], and NMG found in [9–12].

The layout of the paper is as follows: In Sec. II, we recapitulate the algebraic classification of curvature in three dimensions. In Sec. III, we derive the quadratic form of the TMG and NMG equations. Then, we give our main result about $f(R_{\mu\nu})$ theories as a theorem. Section IV is the bulk of the paper where we discuss the solutions of the $f(R_{\mu\nu})$ theories. In Sec. V, as an application, we find the solutions of the quadratic and cubic curvature gravity and BINMG. In appendixes, we give some relevant variations and the field equations of the cubic curvature theory.

II. ALGEBRAIC CLASSIFICATION OF CURVATURE IN THREE DIMENSIONS

In searching for exact solutions, Petrov or Ricci-Segre classification in three dimensions plays an important role [24]. Hence, we briefly review the classification of the exact solutions of TMG given in [23]. The action of TMG with a cosmological constant[3] [14,15]

$$I=-\int d^3x\sqrt{-g}\times\left[R-2\Lambda+\frac{1}{2\mu}\eta^{\alpha\beta\gamma}\Gamma^{\mu}_{\alpha\nu}\left(\partial_{\beta}\Gamma^{\nu}_{\gamma\mu}+\frac{2}{3}\Gamma^{\nu}_{\beta\rho}\Gamma^{\rho}_{\gamma\mu}\right)\right] \quad (1)$$

yields the source-free TMG equations

$$R_{\mu\nu}-\frac{1}{2}g_{\mu\nu}R+\Lambda g_{\mu\nu}+\frac{1}{\mu}C_{\mu\nu}=0, \quad (2)$$

where $C_{\mu\nu}$ is the symmetric, traceless and covariantly conserved Cotton tensor defined as

$$C_{\mu\nu}\equiv\eta_{\mu\alpha\beta}\nabla^{\alpha}\left(R^{\beta}_{\nu}-\delta^{\beta}_{\nu}\frac{R}{4}\right). \quad (3)$$

Here, the Levi-Civita tensor is given as $\eta_{\mu\sigma\rho}=\sqrt{-g}\varepsilon_{\mu\sigma\rho}$ with $\varepsilon_{012}=+1$ and $g\equiv\det[g_{\mu\nu}]$. Taking the trace of (2) gives $R=6\Lambda$. Therefore, the Cotton tensor in TMG becomes $C_{\mu\nu}=\eta_{\mu\sigma\beta}\nabla^{\sigma}R^{\beta}_{\nu}$. Using this, the field equations (2) become

$$R_{\mu\nu}-\frac{1}{3}g_{\mu\nu}R+\frac{1}{\mu}\eta_{\mu\alpha\beta}\nabla^{\alpha}R^{\beta}_{\nu}=0, \quad (4)$$

and defining the traceless-Ricci tensor $S_{\mu\nu}\equiv R_{\mu\nu}-\frac{1}{3}g_{\mu\nu}R$ further reduces the field equations to

$$\mu S_{\mu\nu}=-C_{\mu\nu}. \quad (5)$$

Classification of three-dimensional spacetimes can be done using either the eigenvalues and the eigenvectors of the up-down Cotton tensor (C^{μ}_{ν}) [25] (in analogy with the four-dimensional Petrov classification of the Weyl tensor) or the traceless-Ricci tensor (S^{μ}_{ν}) (in analogy with the Segre classification). Since C^{μ}_{ν} and S^{μ}_{ν} are related through (5), for solutions of TMG Segre and Petrov classifications coincide. As noted in [23], to determine the eigenvalues of S^{ν}_{μ} and their algebraic multiplicities one can compute the two scalar invariants

$$I\equiv S^{\nu}_{\mu}S^{\mu}_{\nu},\qquad J\equiv S^{\sigma}_{\mu}S^{\mu}_{\nu}S^{\nu}_{\sigma}. \quad (6)$$

Petrov-Segre Types O, N, and III satisfy $I=J=0$, while Types D_t, D_s, and II satisfy $I^3=6J^2\neq 0$. Finally, the most general types I_R and I_C satisfy $I^3>6J^2$ and $I^3<6J^2$, respectively.

For Type-N spacetimes, the canonical form of the traceless-Ricci tensor is

$$S_{\mu\nu}=\rho\xi_{\mu}\xi_{\nu}, \quad (7)$$

where ξ_{μ} is a null Killing vector and ρ is a scalar function [9]. On the other hand, Type-D spacetimes split into two types that are denoted as D_t for which the eigenvector of the traceless-Ricci tensor is timelike and D_s for which the eigenvector is spacelike. For both types, the traceless-Ricci tensor takes the form

$$S_{\mu\nu}=p\left(g_{\mu\nu}-\frac{3}{\sigma}\xi_{\mu}\xi_{\nu}\right), \quad (8)$$

where p is a scalar function and ξ_{μ} is a timelike or spacelike vector normalized as $\xi_{\mu}\xi^{\mu}\equiv\sigma=\pm 1$.

In this work, we focus on CSI Type-N and Type-D spacetimes. Type-N spacetimes are CSI if the curvature scalar is constant, while a Type-D spacetime becomes CSI if R and the scalar function p in (8) are constants.[4]

III. A METHOD TO GENERATE SOLUTIONS OF $f(R_{\mu\nu})$ THEORIES

In order to state our main result, first we need to discuss the form of the field equations of TMG in the quadratic form and the field equations of NMG for Type-N and Type-D metrics. One can put (4) in a second order (wavelike) equation in the Ricci tensor as follows. Multiplying (4) with $\eta^{\mu}{}_{\sigma\rho}$ and using

$$\eta^{\mu\alpha\beta}\eta_{\mu\sigma\rho}=-\left(\delta^{\alpha}_{\sigma}\delta^{\beta}_{\rho}-\delta^{\alpha}_{\rho}\delta^{\beta}_{\sigma}\right), \quad (9)$$

[3]Our signature convention is $(-,+,+)$ which is opposite of the original TMG paper; therefore, to account for the "wrong" sign Einstein-Hilbert term, we need to put an overall minus sign to the action. Note that the sign of μ is undetermined in TMG. Furthermore, the overall gravity-matter coupling in the TMG action is taken to be 1, that is, $\kappa=1$, in order to reduce the number of parameters.

[4]As we will demonstrate below, the independent scalar invariants of a three-dimensional spacetime are R, $S^{\mu}_{\nu}S^{\nu}_{\mu}$, $S^{\mu}_{\rho}S^{\nu}_{\mu}S^{\rho}_{\nu}$ which satisfy $S^{\mu}_{\nu}S^{\nu}_{\mu}=6p^2$, $S^{\mu}_{\rho}S^{\nu}_{\mu}S^{\rho}_{\nu}=-6p^3$ for Type-D spacetimes requiring constancy of p.

then, taking the divergence of the resultant equation, one arrives at the desired equation

$$\Box R_{\mu\nu} = \mu^2(R_{\mu\nu} - 2\Lambda g_{\mu\nu}) + 3R_{\mu\lambda}R^{\lambda}_{\nu} - g_{\mu\nu}R^{\rho}_{\sigma}R^{\sigma}_{\rho} - \frac{3}{2}RR_{\mu\nu} + \frac{1}{2}g_{\mu\nu}R^2, \tag{10}$$

whose $\Lambda = 0$ version was given in [14,15]. In fact, in the spirit of [14,15], one can get the same result with the help of the operator

$$\mathcal{O}_{\mu\nu}{}^{\lambda\sigma}(\mu) \equiv \delta^{\lambda}_{\mu}\delta^{\sigma}_{\nu} - \frac{1}{2}g_{\mu\nu}g^{\lambda\sigma}\left(1 - \frac{2\Lambda}{R}\right) + \frac{1}{\mu}\eta_{\mu}{}^{\alpha\beta}\left(\delta^{\lambda}_{\beta}\delta^{\sigma}_{\nu} - \frac{1}{4}g^{\lambda\sigma}g_{\nu\beta}\right)\nabla_{\alpha}, \tag{11}$$

namely, $\mu^2\mathcal{O}_{\alpha\beta}{}^{\mu\nu}(-\mu)\mathcal{O}_{\mu\nu}{}^{\lambda\sigma}(\mu)R_{\lambda\sigma} = 0$ reproduces (10). It is more transparent to write the quadratic TMG equation as a pure trace and a traceless part as

$$R = 6\Lambda, \tag{12}$$

$$(\Box - \mu^2 - 3\Lambda)S_{\mu\nu} = 3S_{\mu\rho}S^{\rho}_{\nu} - g_{\mu\nu}S_{\sigma\rho}S^{\sigma\rho}, \tag{13}$$

where $S_{\mu\nu}$ is the traceless-Ricci tensor $S_{\mu\nu} \equiv R_{\mu\nu} - \frac{1}{3}g_{\mu\nu}R$. It is important to note that every solution of TMG (2) solves (13), but not every solution of the latter solves the former. For Type-N spacetimes, (13) reduces to

$$\Box S_{\mu\nu} = (\mu^2 + 3\Lambda)S_{\mu\nu}, \tag{14}$$

while for Type-D spacetimes, it becomes

$$\Box S_{\mu\nu} = (\mu^2 + 3\Lambda - 3p)S_{\mu\nu}. \tag{15}$$

Besides using the solutions of TMG, we will also use the solutions of NMG in order to find solutions to the $f(R^{\mu}_{\nu})$ type theories. Hence, let us write the field equations of NMG. In order to create a parametrization difference between NMG and the generic quadratic curvature theory that we study, and to directly use the results given in [10–12], it is better to prefer the parametrization of NMG used in these works. Then, let us take the action of NMG as[5]

$$I_{\text{NMG}} = -\frac{1}{16\pi G}\int d^3x\sqrt{-g}\left[R - 2\lambda - \frac{1}{m^2}\left(R^{\mu}_{\nu}R^{\nu}_{\mu} - \frac{3}{8}R^2\right)\right]. \tag{16}$$

One can get the field equations of NMG by using the field equations of generic cubic curvature gravity given in (B3) and (B4) as

$$S_{\mu\nu}S^{\mu\nu} + m^2R - \frac{1}{24}R^2 = 6m^2\lambda, \tag{17}$$

and

$$\left(\Box - m^2 - \frac{5}{12}R\right)S_{\mu\nu} = 4\left(S_{\mu\rho}S^{\rho}_{\nu} - \frac{1}{3}g_{\mu\nu}S_{\sigma\rho}S^{\sigma\rho}\right) + \frac{1}{4}\left(\nabla_{\mu}\nabla_{\nu} - \frac{1}{3}g_{\mu\nu}\Box\right)R. \tag{18}$$

Here, the trace field equation is in the form given in [11,12], while the traceless field equation corresponds to the equation

$$(\not{D}^2 - m^2)S_{\mu\nu} = T_{\mu\nu}, \tag{19}$$

where the operator $\not{D}$ is defined through its action on a symmetric tensor $\Phi_{\mu\nu}$ as

$$\not{D}\Phi_{\mu\nu} \equiv \frac{1}{2}(\eta_{\mu}{}^{\alpha\beta}\nabla_{\beta}\Phi_{\nu\alpha} + \eta_{\nu}{}^{\alpha\beta}\nabla_{\beta}\Phi_{\mu\alpha}), \tag{20}$$

and

$$T_{\mu\nu} = S_{\mu\rho}S^{\rho}_{\nu} - \frac{1}{3}g_{\mu\nu}S_{\sigma\rho}S^{\sigma\rho} - \frac{R}{12}S_{\mu\nu}. \tag{21}$$

It is important to note that two forms of the traceless field equations of NMG, (18) and (19), are equivalent whether or not the scalar curvature R is constant. Now, let us write the field equations of NMG for Type-N spacetimes. The trace equation (17) becomes

$$m^2R - \frac{1}{24}R^2 = 6m^2\lambda, \tag{22}$$

which implies that the scalar curvature is constant. The traceless field equation (18) takes the form

$$\Box S_{\mu\nu} = \left(m^2 + \frac{5}{12}R\right)S_{\mu\nu} \tag{23}$$

after using the constancy of the scalar curvature. It is easy to see that Eqs. (14) and (23) are the same equations with different parametrizations of the constant parts which are related as

$$\mu^2 = m^2 - \frac{R}{12}. \tag{24}$$

By using this observation, the Type-N solutions of NMG that are based on the Type-N solutions of TMG are found in [10,11]. On the other hand, once the Type-D ansatz is inserted into the NMG equations (17) and (18), one obtains

$$6p^2 + m^2R - \frac{1}{24}R^2 = 6m^2\lambda, \tag{25}$$

and

$$\left(\Box - m^2 - \frac{5}{12}R + 4p\right)S_{\mu\nu} = \frac{1}{4}\left(\nabla_{\mu}\nabla_{\nu} - \frac{1}{3}g_{\mu\nu}\Box\right)R. \tag{26}$$

Since we are interested in the constant scalar curvature solutions of the $f(R^{\mu}_{\nu})$ theory, implementing this assumption in (25) implies that p is also a constant and (26) takes the form

[5]Notice that we introduce an overall minus sign to the action, with the assumption that $G > 0$. Because, in order that NMG defines a unitary theory, one needs the wrong sign Einstein-Hilbert term.

$$\Box S_{\mu\nu} = \left(m^2 + \frac{5}{12}R - 4p\right)S_{\mu\nu}. \tag{27}$$

As in the Type-N case, (27) and the traceless field equation of TMG for Type-D spacetimes given in (15) are the same equation with different parametrizations which are related by

$$\mu^2 = m^2 - \frac{R}{12} - p. \tag{28}$$

This observation led to the Type-D solutions of NMG based on the Type-D solutions of TMG [9,10].

After the above discussion of the field equations of TMG and NMG, let us focus on the $f(R_{\mu\nu})$ theory. As we will show in the next section, for the CSI Type-N and Type-D spacetimes, the trace field equation of generic $f(R_{\mu\nu})$ theory determines the constant scalar curvature in terms of the parameters of the theory, while the traceless field equations reduce to the form

$$(\Box - c)S_{\mu\nu} = 0, \tag{29}$$

where c is a function of the parameters of the theory.[6] This fact leads us to our main solution inheritance result:

Theorem: A Type-N or Type-D solution of TMG or NMG that has constant scalar curvature generates a solution of generic $f(R_{\mu\nu})$ theory provided that the relations between the parameters of the corresponding theories, which are obtained by putting the solution of TMG or NMG as an ansatz into the field equations of the $f(R_{\mu\nu})$ theory, are satisfied.

Proof: For Type-N and Type-D spacetimes of constant scalar curvature, the traceless field equations of TMG, NMG, and generic $f(R_{\mu\nu})$ theory take the same wavelike equation for $S_{\mu\nu}$ given in (14) and (15), (23) and (27), and (29), respectively. Hence, the main relation between the parameters of the corresponding theories can be obtained by simply replacing the term $\Box S_{\mu\nu}$ in the traceless field equation of the $f(R_{\mu\nu})$ theory by use of the field equations of TMG or NMG. Besides, for Type-D solutions of TMG (NMG), a specific relation between μ^2 (m^2), the scalar curvature, and p appearing in (8) needs to be satisfied. Finally, the trace field equation of the $f(R_{\mu\nu})$ theory determines the scalar curvature in terms of the theory parameters. Provided that this set of equations relating the parameters in the corresponding theories is solved, one manages to map the constant scalar curvature Type-N and Type-D solutions of TMG or NMG to solutions of the $f(R_{\mu\nu})$ theory through these relations.

In the following sections, for the $f(R_{\mu\nu})$ theory, we give the explicit forms of the relations mentioned above and apply the results to quadratic and cubic curvature theories and BINMG.

[6] For Type-D spacetimes, c also depends on p appearing in (8).

IV. SOLUTIONS OF $f(R_{\mu\nu})$ THEORIES IN THREE DIMENSIONS

In three dimensions, the Riemann tensor does not carry more information than the Ricci tensor; hence a generic higher curvature theory is just built on the contractions of the Ricci tensor as

$$I = \int d^3x\sqrt{-g}\left[\frac{1}{\kappa}(R - 2\Lambda_0) + \sum_{n=2}^{\infty}\sum_{\substack{i=0\\ i\neq 1}}^{n}\sum_{j} a_{ni}^{j}(R^{\mu}_{\nu})^{i}_{j}R^{n-i}\right], \tag{30}$$

where the superscript i in $(R^{\mu}_{\nu})^{i}_{j}$ represents the number of Ricci tensors in the term, while the summation on the subscript j represents the number of possible ways to contract the i number of Ricci tensors. Each higher curvature combination has a different coupling denoted by a_{ni}^{j}. In the summation over i, the value of 1 is not allowed, simply because it just yields the scalar curvature upon contraction which is accounted for. For a given i, finding the possible ways of contracting the i number of Ricci tensors is a counting problem of finding the sequences of integers satisfying[7]

$$i = \sum_{r=1}^{r_{\max}} s_r; \qquad s_r \leq s_{r+1}, \qquad s_1 \geq 2. \tag{31}$$

Each number in the sequence represents a scalar form involving that number of Ricci tensors contracted as

$$R^{\mu_1}_{\mu_{s_r}}\prod_{i=2}^{s_r}R^{\mu_i}_{\mu_{i-1}}. \tag{32}$$

As an example, let us discuss the terms appearing at the curvature order $n = 7$. Even though, the example seems cumbersome, it is quite useful to understand the counting problem here. The i summation in (30) consists of the following 7 terms:

$$R^7, \quad (R^{\mu}_{\nu})^2R^5, \quad (R^{\mu}_{\nu})^3R^4, \quad (R^{\mu}_{\nu})^4R^3,$$
$$(R^{\mu}_{\nu})^5R^2, \quad (R^{\mu}_{\nu})^6R, \quad (R^{\mu}_{\nu})^7.$$

For $i = 4, 5, 6, 7$, the possible sequences satisfying (31) are

$i = 4$: (2, 2), (4); $\quad i = 5$: (2, 3), (5);

$i = 6$: (2, 2, 2), (2, 4), (3, 3), (6);

$i = 7$: (2, 2, 3), (2, 5), (3, 4), (7).

For example, for $n = i = 7$, the terms are

[7] For the construction of all possible terms at a given order n, see also [26].

$$R^{\mu_1}_{\mu_2}R^{\mu_2}_{\mu_1}R^{\mu_3}_{\mu_4}R^{\mu_4}_{\mu_3}R^{\mu_5}_{\mu_7}R^{\mu_6}_{\mu_5}R^{\mu_7}_{\mu_6},$$
$$R^{\mu_1}_{\mu_2}R^{\mu_2}_{\mu_1}R^{\mu_3}_{\mu_7}R^{\mu_4}_{\mu_3}R^{\mu_5}_{\mu_4}R^{\mu_6}_{\mu_5}R^{\mu_7}_{\mu_6},$$
$$R^{\mu_1}_{\mu_3}R^{\mu_2}_{\mu_1}R^{\mu_3}_{\mu_2}R^{\mu_4}_{\mu_7}R^{\mu_5}_{\mu_4}R^{\mu_6}_{\mu_5}R^{\mu_7}_{\mu_6},$$
$$R^{\mu_1}_{\mu_7}R^{\mu_2}_{\mu_1}R^{\mu_3}_{\mu_2}R^{\mu_4}_{\mu_3}R^{\mu_5}_{\mu_4}R^{\mu_6}_{\mu_5}R^{\mu_7}_{\mu_6}.$$

What is important to realize is that at each order n, there is only one term which cannot be constructed as a multiplication of the terms that already appear at the lower orders compared to n. This term is $(R^\mu_\nu)^n$ with the contraction sequence (n) that is

$$R^{\mu_1}_{\mu_n}R^{\mu_2}_{\mu_1}R^{\mu_3}_{\mu_2}\dots R^{\mu_n}_{\mu_{n-1}}. \tag{33}$$

However, it is shown in [26] that for $n > 3$ the term (33) can be written as a sum of the other terms appearing in order n by use of the Schouten identities:

$$\delta^{\mu_1\mu_2\dots\mu_n}_{\nu_1\nu_2\dots\nu_n}R^{\nu_1}_{\mu_1}R^{\nu_2}_{\mu_2}\dots R^{\nu_n}_{\mu_n} = 0, \qquad n > 3, \tag{34}$$

where $\delta^{\mu_1\mu_2\dots\mu_n}_{\nu_1\nu_2\dots\nu_n}$ is the generalized Kronecker delta with the definition

$$\delta^{\mu_1\dots\mu_{2n}}_{\nu_1\dots\nu_{2n}} \equiv \det\begin{vmatrix} \delta^{\mu_1}_{\nu_1} & \dots & \delta^{\mu_{2n}}_{\nu_1} \\ \vdots & \ddots & \vdots \\ \delta^{\mu_1}_{\nu_{2n}} & \dots & \delta^{\mu_{2n}}_{\nu_{2n}} \end{vmatrix}. \tag{35}$$

The basis for the Schouten identities is the simple fact that in three dimensions a totally antisymmetric tensor having a rank higher than 3 is identically zero. Therefore, for $n > 3$, the new term appearing at order n given in (33) can be written as a sum of the terms which involve n curvature forms and are multiplications of the terms that already appear at the lower orders. This fact implies that the terms R, $R^\mu_\nu R^\nu_\mu$, $R^\mu_\rho R^\nu_\mu R^\rho_\nu$ are the only independent curvature combinations, and every other term that can be constructed by any kind of contraction of any number of Ricci tensors can be obtained as a function of these three terms [26]. Therefore, a higher curvature gravity action of the form $f(R^\mu_\nu)$ either given in a series expansion or in a closed form can be put in a form $f(R^\mu_\nu) = F(R, R^\mu_\nu R^\nu_\mu, R^\mu_\rho R^\nu_\mu R^\rho_\nu)$.

As revealed in the previous sections, working with the traceless-Ricci tensor instead of the Ricci tensor at the equation of motion level simplifies the computations. Let us consider this change at the action level, and obtain the field equations in terms of the traceless-Ricci tensor directly. With this change every observation made in the previous paragraphs remains the same with one simplification: the Schouten identity written in terms of the traceless-Ricci tensor

$$0 = \delta^{\mu_1\mu_2\dots\mu_n}_{\nu_1\nu_2\dots\nu_n}S^{\nu_1}_{\mu_1}S^{\nu_2}_{\mu_2}\dots S^{\nu_n}_{\mu_n}, \qquad n > 3, \tag{36}$$

involves a fewer number of terms than (34) due to vanishing trace of $S_{\mu\nu}$.

Now, let us study the field equation of higher curvature gravity theories. With the hindsight gained in the previous paragraphs, it is sufficient and convenient to study the action

$$I = \int d^3x\sqrt{-g}F(R, A, B), \tag{37}$$

where

$$A \equiv S^\mu_\nu S^\nu_\mu, \qquad B \equiv S^\mu_\rho S^\nu_\mu S^\rho_\nu,$$

and F is either a power series expansion in (R, A, B) or an analytic function in these variables. It is worth restating the arguments on this choice: studying this action is sufficient since any higher curvature action which involves just the scalars constructed from only R^μ_ν, and not its derivatives, can be put in this form; on the other hand, it is convenient to study a generic form of a higher curvature action in (R, A, B) because we aim to figure out the general structure of the Type-N and Type-D solutions whose analysis becomes easy by using the canonical form of the traceless-Ricci tensor. The variation of the action (37) has the form

$$\delta I = \int d^3x\sqrt{-g}\Big(F_R\delta R + F_A\delta A + F_B\delta B - \frac{1}{2}g_{\mu\nu}F\delta g^{\mu\nu}\Big), \tag{38}$$

where $F_R \equiv \frac{\partial F}{\partial R}$, and F_A, F_B are defined similarly. Using the δR, δA, and δB results given in the appendix, the field equations for the action (37) become

$$\begin{aligned} &-\frac{1}{2}g_{\mu\nu}F + 2F_A S^\rho_\mu S_{\rho\nu} + 3F_B S^\rho_\mu S_{\rho\sigma}S^\sigma_\nu + \Big(\Box + \frac{2}{3}R\Big)\Big(F_A S_{\mu\nu} + \frac{3}{2}F_B S^\rho_\mu S_{\rho\nu}\Big) \\ &\quad + \Big(g_{\mu\nu}\Box - \nabla_\mu\nabla_\nu + S_{\mu\nu} + \frac{1}{3}g_{\mu\nu}R\Big)(F_R - F_B S^\rho_\sigma S^\sigma_\rho) - 2\nabla_\alpha\nabla_{(\mu}\Big(S^\alpha_{\nu)}F_A + \frac{3}{2}S^\rho_{\nu)}S^\alpha_\rho F_B\Big) \\ &\quad + g_{\mu\nu}\nabla_\alpha\nabla_\beta\Big(F_A S^{\alpha\beta} + \frac{3}{2}F_B S^{\alpha\rho}S^\beta_\rho\Big) = 0. \end{aligned} \tag{39}$$

A simple observation is that the Type-O spacetimes (for which $S_{\mu\nu} = 0$) with constant scalar curvature satisfy the field equation

$$\frac{3}{2}F - RF_R = 0. \tag{40}$$

Furthermore, note that A and B are zero for Type-N spacetimes, and they are proportional to p^2 and p^3, respectively, for Type-D spacetimes. Therefore, F, F_R, F_A, F_B are functions of R for Type-N spacetimes, while they are functions of R and p for Type-D spacetimes.

Now, let us study the Type-N and Type-D solutions of the $f(R^\mu_\nu)$ gravity which are also solutions of the cosmological TMG or NMG. In finding these solutions, we will assume that the spacetime is CSI. This assumption implies that the scalar curvature is constant in addition to the constancy of p for Type-D spacetimes. Without such an assumption, one cannot proceed unless the explicit form of F, that is, the action, is given.

A. Type-N solutions

Recall that for Type-N spacetimes contractions of two and more traceless-Ricci tensors vanish; therefore, for such spacetimes (39) becomes

$$\begin{aligned} -\frac{1}{2}g_{\mu\nu}F &+ \Big(\Box + \frac{2}{3}R\Big)(F_A S_{\mu\nu}) \\ &+ \Big(g_{\mu\nu}\Box - \nabla_\mu\nabla_\nu + S_{\mu\nu} + \frac{1}{3}g_{\mu\nu}R\Big)F_R \\ &- 2\nabla_\alpha\nabla_{(\mu}(S^\alpha_{\nu)}F_A) + g_{\mu\nu}\nabla_\alpha\nabla_\beta(F_A S^{\alpha\beta}) = 0. \end{aligned} \tag{41}$$

Constancy of the scalar curvature R implies that F, F_R, F_A, F_B are all constants, since they only depend on R for Type-N spacetimes. Besides, one has the Bianchi identity $\nabla_\mu S^\mu_\nu = \frac{1}{6}\nabla_\nu R = 0$ which further simplifies (41) to

$$\Big(\frac{1}{3}RF_R - \frac{1}{2}F\Big)g_{\mu\nu} + \Big(F_A\Box - \frac{1}{3}RF_A + F_R\Big)S_{\mu\nu} = 0. \tag{42}$$

Since $S_{\mu\nu}$ is traceless, the field equations split into two parts as

$$\frac{3}{2}F - RF_R = 0, \tag{43}$$

$$\Big(F_A\Box - \frac{1}{3}RF_A + F_R\Big)S_{\mu\nu} = 0, \tag{44}$$

which are the trace and the traceless field equations of the higher curvature gravity theory for Type-N spacetimes with constant curvature. The first equation determines the scalar curvature, and the second equation is of the form $(\Box - c(R; a^j_{ni}))S_{\mu\nu} = 0$, where $c(R; a^j_{ni})$ is a constant depending on R and the parameters of the $f(R^\mu_\nu)$ theory and does not vanish generically. Even though we have reduced the complicated field equations of the generic $f(R^\mu_\nu)$ theory to a Klein-Gordon type equation for $S_{\mu\nu}$, it is still a highly complicated nonlinear equation for the metric and without further assumptions such as the existence of symmetries it would be hard to find explicit solutions. But the state of affairs is not that bleak, as we will lay out below, the field equations of TMG (in the quadratic form) and NMG also reduce to Klein-Gordon type equations for $S_{\mu\nu}$ for Type-N spacetimes.[8] Such solutions in these theories have been studied before. In [11], the Type-N solutions for the equation $(\Box - c(R;\mu^2))S_{\mu\nu} = 0$ with constant curvature is studied where the form of $c(R;\mu^2)$ is specifically the one corresponding to the quadratic form of the TMG field equations.

Now, let us discuss the solutions based on the solutions of TMG and NMG, separately, and elaborate on the relation between them.

1. Solutions based on TMG

The Type-N solutions of TMG satisfy the field equations (12) and (14). Then, requiring that the Type-N solutions of $f(R^\mu_\nu)$ gravity are also solutions of TMG, the field equations (43) and (44) take the forms

$$F - 4\Lambda F_R = 0, \tag{45}$$

and

$$\mu^2 = -\Big(\frac{F_R}{F_A} + \Lambda\Big). \tag{46}$$

Generically, (45) is not an algebraic equation. If it is solved for the unknown Λ, its solution together with (46) fixes Λ and μ^2 in terms of the parameters of the higher curvature theory. Once these two equations are satisfied Type-N solutions of TMG compiled in [23,27] also solve the $f(R^\mu_\nu)$ theory.

2. Solutions based on NMG

The field equations of NMG for Type-N spacetimes reduce to (22) and (23). One should recall that the traceless field equations of TMG (14) and NMG (23) are the same equations with different parametrizations related as (24). In [9,11], the Type-N solutions of NMG are parametrized by R and μ.[9] Therefore, in order to find the Type-N solutions of the $f(R^\mu_\nu)$ theory which are also solutions of NMG, (45) and (46) are again the equations which need to be satisfied. The Type-N solutions of TMG that we used in the previous section satisfy the traceless field equation of TMG

$$\eta_{\mu\alpha\beta}\nabla^\alpha S^\beta_\nu + \mu S_{\mu\nu} = 0, \tag{47}$$

besides the second order equation (14). The general Type-N solution of NMG given in [11] includes the TMG based solutions as special limits in addition to the solutions which only solve the quadratic equation (14). Thus, once the Type-N solutions of the $f(R^\mu_\nu)$ theory with NMG origin are found, the Type-N solutions based on TMG are also obtained by considering the limits given in [11]. Note that

[8]This is the observation which yields the Type-N solutions of NMG which are also solutions of TMG found in [9,10].

[9]In fact, (14) is the equation solved in [11] with the definition $\nu^2 \equiv -R/6$.

as shown in [10,11], there are two classes of Type-N solutions of NMG depending on whether the eigenvector $\xi_\nu = \partial_\nu$ of $S_{\mu\nu}$ is a Killing vector or not. Here, we have covered both of these Type-N solutions.

B. Type-D solutions

First of all, we just employ the canonical form of the traceless-Ricci tensor for Type-D spacetimes given in (8) in the field equations of the $f(R^\mu_\nu)$ theory given in (39). In the equations, there are rank (0, 2) tensors formed by the contractions of two and three traceless-Ricci tensors. With the use of (8), one can show that these forms are just linear combinations of the metric and $S_{\mu\nu}$ as

$$\begin{aligned} S^\rho_\mu S_{\rho\nu} &= p(2pg_{\mu\nu} - S_{\mu\nu}), \\ S^\rho_\mu S_{\rho\sigma} S^\sigma_\nu &= p^2(3S_{\mu\nu} - 2pg_{\mu\nu}). \end{aligned} \tag{48}$$

Therefore, for Type-D spacetimes, the rank (0, 2) tensors that should appear in the equations of motion of the $f(R^\mu_\nu)$ theory are just the metric and the traceless-Ricci tensor, and consequently, (39) takes the form

$$\begin{aligned} &\left(-\frac{1}{2}F + 4p^2F_A - 6p^3F_B\right)g_{\mu\nu} + (-2pF_A + 9p^2F_B)S_{\mu\nu} \\ &\quad + \left(\Box + \frac{2}{3}R\right)\left[3p^2F_B g_{\mu\nu} + \left(F_A - \frac{3}{2}pF_B\right)S_{\mu\nu}\right] \\ &\quad + \left(g_{\mu\nu}\Box - \nabla_\mu\nabla_\nu + S_{\mu\nu} + \frac{1}{3}g_{\mu\nu}R\right)(F_R - 6p^2F_B) \\ &\quad - 2\nabla_\alpha\nabla_{(\mu}\left[\delta^\alpha_{\nu)}3p^2F_B + S^\alpha_{\nu)}\left(F_A - \frac{3}{2}pF_B\right)\right] \\ &\quad + g_{\mu\nu}\nabla_\alpha\nabla_\beta\left[3p^2F_B g^{\alpha\beta} + \left(F_A - \frac{3}{2}pF_B\right)S^{\alpha\beta}\right] = 0. \end{aligned} \tag{49}$$

If R and p are assumed to be constant (note that p should be a constant for TMG, and the constancy of R implies the constancy of p for the NMG case), then F, F_R, F_A, F_B should also be constant due to the fact that they just depend on R and p. Then, (49) reduces to

$$\begin{aligned} 0 = &\left[-\frac{1}{2}F + \frac{1}{3}RF_R + 4p^2\left(F_A - \frac{3}{2}pF_B\right)\right]g_{\mu\nu} \\ &+ \left[F_R + \left(F_A - \frac{3}{2}pF_B\right)\left(\Box - \frac{1}{3}R + 4p\right)\right]S_{\mu\nu}, \end{aligned} \tag{50}$$

where the Bianchi identity was also used. Since $S_{\mu\nu}$ is traceless, the field equations split into two parts as

$$\frac{3}{2}F - RF_R - 6p^2(2F_A - 3pF_B) = 0, \tag{51}$$

$$\left[F_R + \left(F_A - \frac{3}{2}pF_B\right)\left(\Box - \frac{1}{3}R + 4p\right)\right]S_{\mu\nu} = 0. \tag{52}$$

They are the trace and the traceless field equations of the $f(R^\mu_\nu)$ theory for the Type-D spacetimes with constant R and p. In (51), the unknowns are p and R; therefore, unlike the case of the Type-N spacetimes, the trace field equation does not yield a solution for R. On the other hand, as in the case of Type-N spacetimes, the only operator in the traceless field equation is the d'Alembertian, $\Box$, so the equation is in the form

$$[\Box - c(R, p; a^j_{ni})]S_{\mu\nu} = 0, \tag{53}$$

where $c(R, p; a^j_{ni})$ is a constant. For Type-D spacetimes of constant curvature, the traceless field equations of TMG (in the quadratic form) and NMG are also in the same Klein-Gordon form where the functional dependence of c is $c(R, p; \mu^2)$ in the TMG case and $c(R, p; m^2)$ in the NMG case. However, the set $(R, p; \mu^2)$ of the TMG case and the set $(R, p; m^2)$ of the NMG case are not independent, but satisfy specific algebraic relations in order to yield Type-D solutions. We use these algebraic relations to put the traceless field equation of TMG or NMG in the form $[\Box - c(R, p)]S_{\mu\nu} = 0$. Then, assuming that the Type-D solutions of either TMG or NMG are also Type-D solutions of the $f(R^\mu_\nu)$ theory, one can replace $\Box S_{\mu\nu}$ with the term $c(R, p)S_{\mu\nu}$. The resulting equation together with (51) constitutes a coupled set of equations that needs to be satisfied in order to have constant curvature Type-D solutions of the $f(R^\mu_\nu)$ theory which are also solutions of TMG or NMG. Now, let us discuss the Type-D solutions based on TMG and NMG, and their relation.

1. Solutions based on TMG

For Type-D solutions of the cosmological TMG, as shown in [28,29], the function p appearing in (8) is a constant and the vector ξ^μ should be a Killing vector satisfying

$$\nabla_\mu \xi_\nu = \frac{\mu}{3}\eta_{\mu\nu\rho}\xi^\rho, \tag{54}$$

where, in order to have a solution, μ should be related to p and Λ as

$$\mu^2 = 9(p - \Lambda). \tag{55}$$

Type-D solutions of TMG automatically satisfy the traceless field equation of TMG in the quadratic form given in (15) which becomes

$$[\Box + 6(\Lambda - p)]S_{\mu\nu} = 0, \tag{56}$$

with the help of (55). If one requires that the Type-D solutions of the $f(R^\mu_\nu)$ theory are also solutions of TMG, then by using (56) and the trace field equation of TMG, that is, $R = 6\Lambda$, in the field equations of the $f(R^\mu_\nu)$ theory for Type-D spacetimes given in (51) and (52), one arrives at

$$F - 4\Lambda F_R - 4p^2(2F_A - 3pF_B) = 0, \tag{57}$$

and

$$F_R - (2F_A - 3pF_B)(4\Lambda - 5p) = 0. \tag{58}$$

This coupled set of equations determines p and Λ in terms of the parameters of the $f(R^\mu_\nu)$ theory. One should keep in mind that the solutions of this set should satisfy (55), or, in other words, they determine μ^2. Once Λ and μ^2 are found, their use in the Type-D solutions of TMG compiled in [23] which are parametrized by Λ and μ^2 yields the Type-D solutions of the $f(R^\mu_\nu)$ theory.

2. Solutions based on NMG

As we discussed in Sec. III, for constant scalar curvature Type-D spacetimes, the field equations of NMG take the forms (25) and (27), and the latter equation is nothing but the reparametrized version of the traceless field equation of TMG for Type-D spacetimes given in (15). The constancy of R implies the constancy of p via (25) which in turn indicates that the constant scalar curvature solutions of NMG are CSI spacetimes.

Like the Type-D solutions of TMG, (27) is solved if and only if the parameters p, R, and m^2 are related in a specific way. Equations (55) and (28) yield

$$p = \frac{m^2}{10} + \frac{17}{120}R. \tag{59}$$

If (59) is satisfied, then there are Type-D solutions of NMG which are also solutions of TMG with the parameters given below by (87) and (88) after using $\alpha = -3\beta/8$, $m^2 = -\frac{1}{\kappa\beta}$ [12]. Furthermore, there are also Type-D solutions which are exclusively solutions of NMG, but not solutions of TMG. These solutions are separated into two classes differing with respect to the relations satisfied by the parameters of NMG and its Type-D solution. This follows whether ξ_μ is a hypersurface orthogonal Killing vector or a covariantly divergence-free vector, not a Killing vector [12]. For the covariantly divergence-free vector case, the parameters should be related as

$$p = -\frac{R}{3} = -\frac{4}{15}m^2, \qquad \lambda = \frac{m^2}{5}, \tag{60}$$

while for the case of a hypersurface orthogonal Killing vector, the parameters are related as

$$p = \frac{R}{6} = \frac{2}{3}m^2, \qquad \lambda = m^2. \tag{61}$$

As it can be observed from (60) and (61), these Type-D solutions provided in [12] are uniquely parametrized by m^2.

By requiring that the Type-D solutions of the $f(R^\mu_\nu)$ theory to be also solutions of NMG, one can use (27) to reduce the traceless field equation of the $f(R^\mu_\nu)$ theory given in (52) to

$$F_R + \left(F_A - \frac{3}{2}pF_B\right)\left(m^2 + \frac{1}{12}R\right) = 0. \tag{62}$$

Then, (51) and (62) constitute the equations that should be solved in order to write the parameters of the Type-D solutions of NMG in terms of the parameters of the $f(R^\mu_\nu)$ theory. Besides, the parameters p, R, and m^2 appearing in (51) and (62) need to satisfy one of the equations (59), (60), or (61). If one chooses to eliminate m^2 in favor of p and R by using (59), then (51) and (62) reduce to (57) and (58) of the TMG case. Therefore, one obtains the same Type-D solutions discussed in the previous section, if p, R, and m^2 satisfy (59). On the other hand, if these parameters satisfy (60) or (61), then (51) and (62) reduce either to

$$F = 0, \qquad F_R + \frac{2}{3}R(2F_A + RF_B) = 0 \tag{63}$$

or to

$$F = 0, \qquad F_R + \frac{R}{3}\left(F_A - \frac{1}{4}RF_B\right) = 0. \tag{64}$$

In (63) and (64), one of the equations can be used in order to determine the constant curvature scalar R in terms of the parameters of the $f(R^\mu_\nu)$ theory, while the other equation is a constraint on the parameters of the $f(R^\mu_\nu)$ theory, as the parameter λ in NMG was constrained to take a specific value in terms of m^2. Once R is determined, it can be used to determine m^2 in terms of the parameters of the $f(R^\mu_\nu)$ theory, and therefore, one obtains the Type-D solutions of the $f(R^\mu_\nu)$ theory since, as we noted, m^2 is the unique parameter appearing in the Type-D solutions of NMG belonging to these two cases represented with Eqs. (60) and (61).

V. APPLICATIONS

We are now ready to employ the results obtained for the general case of $f(R_{\mu\nu})$ theories in finding the constant scalar curvature Type-N and Type-D solutions to general quadratic and general cubic curvature gravity theories and the BINMG theory. We have studied the solutions of the quadratic curvature gravity as it is the simplest case for which the Type-N and Type-D solutions can be obtained by mapping the corresponding solutions of TMG and NMG. Furthermore, the quadratic curvature gravity is a simple setting for which the explicit study of the new solutions directly through the field equations is rather instructive. On the other hand, the solutions for the generic cubic curvature gravity case naturally provide new solutions to the cubic curvature extension of NMG which was introduced by using the holography ideas [3].[10] Like the cubic curvature case, one can also use the results of the $f(R_{\mu\nu})$ theory in order to study the solutions of all the higher curvature extensions of NMG based on the holography ideas given in [3,26]. Finally, BINMG [4] is an interesting theory either as an infinite order in curvature extension of NMG which is unitary [5] and has a holographic c function matching that of Einstein's gravity [22] or on its own right as it appears as

[10] Note that this extension also coincides with small curvature expansion of BINMG in the third order [4].

a cutoff independent counterterm to the four-dimensional anti–de Sitter space [30] and as the first example of a unitary Born-Infeld type gravity [31]. We have provided all Type-N solutions of BINMG by using the result of [11], while the Type-D solutions that we have found are constrained to the CSI spacetimes.

When one searches for solutions to a given theory in a standard way, the first thing to do is to obtain the field equations which is often a cumbersome task if the theory involves higher curvature terms. Indeed, the field equations of the cubic curvature gravity [see (B1)] and BINMG (see [22]) are quite involved. Then, preferably the field equations are simplified by a suitable choice of ansatz such as assuming $S_{\mu\nu}$ to be of Type N or Type D as was done here. Finally, the remaining equations, which still have a nonlinear complicated form in metric, are needed to be solved. However, by using the results obtained for generic $f(R_{\mu\nu})$ theory, one bypasses all these complications and obtains the solutions by mapping the already existing solutions of TMG and NMG via rather simple relations between the parameters of the theories.

A. Solutions of quadratic and cubic curvature gravity

Since cubic curvature theories include the quadratic ones, we start with the most general cubic curvature gravity in three dimensions with the action

$$I = \int d^3x\sqrt{-g}\Big[\frac{1}{\kappa}(R-2\Lambda_0) + \alpha R^2 + \beta R^\mu_\nu R^\nu_\mu + \gamma_1 R^\mu_\nu R^\rho_\mu R^\nu_\rho + \gamma_2 R R^\mu_\nu R^\nu_\mu + \gamma_3 R^3\Big]. \tag{65}$$

In order to use the results of the previous section, first we need to rewrite (65) in terms of (R, A, B) as

$$I = \int d^3x\sqrt{-g}\Big[\frac{1}{\kappa}(R-2\Lambda_0) + \Big(\alpha+\frac{\beta}{3}\Big)R^2 + \beta S^\mu_\nu S^\nu_\mu + \gamma_1 S^\mu_\nu S^\rho_\mu S^\nu_\rho + (\gamma_1+\gamma_2)R S^\mu_\nu S^\nu_\mu + \Big(\frac{\gamma_1}{9}+\frac{\gamma_2}{3}+\gamma_3\Big)R^3\Big]. \tag{66}$$

Type-N solutions of the cubic curvature gravity can be found by solving the set (45) and (46) where the terms F, F_R, and F_A for the cubic curvature gravity with the Type-N ansatz have the forms

$$F = \frac{1}{\kappa}(R-2\Lambda_0) + \Big(\alpha+\frac{\beta}{3}\Big)R^2 + \Big(\frac{\gamma_1}{9}+\frac{\gamma_2}{3}+\gamma_3\Big)R^3, \tag{67}$$

$$F_R = \frac{1}{\kappa} + 2\Big(\alpha+\frac{\beta}{3}\Big)R + \Big(\frac{\gamma_1}{3}+\gamma_2+3\gamma_3\Big)R^2, \qquad F_A = \beta + (\gamma_1+\gamma_2)R. \tag{68}$$

After employing these in (45) and (46), one can find the following relation between the parameters of the cubic theory and μ^2, Λ:

$$\mu^2 = -[\beta + 6\Lambda(\gamma_1+\gamma_2)]^{-1}\Big[\frac{1}{\kappa} + \Lambda(12\alpha+5\beta) + 6\Lambda^2(3\gamma_1+7\gamma_2+18\gamma_3)\Big], \tag{69}$$

provided that $\beta + 6\Lambda(\gamma_1+\gamma_2) \neq 0$, and Λ should satisfy

$$\frac{\Lambda-\Lambda_0}{2\kappa} - (3\alpha+\beta)\Lambda^2 - 6(\gamma_1+3\gamma_2+9\gamma_3)\Lambda^3 = 0, \tag{70}$$

whose solutions are not particularly illuminating to depict. Therefore, if μ^2 and Λ of the NMG Type-N solutions [11], which also involve the Type-N solutions of TMG, are tuned with the parameters of the cubic theory according to (69) and (70), then these spacetimes also solve the cubic theory. Furthermore, setting $\gamma_1 = \gamma_2 = \gamma_3 = 0$ yields the Type-N field equations of quadratic curvature gravity whose solutions are given below in (84) and (85).

Now, moving on to the Type-D case, first one needs to calculate F, F_R, F_A, F_B from (66) for the Type-D spacetime ansatz which become

$$\begin{aligned} F &= \frac{1}{\kappa}(R-2\Lambda_0) + \Big(\alpha+\frac{\beta}{3}\Big)R^2 + \Big(\frac{\gamma_1}{9}+\frac{\gamma_2}{3}+\gamma_3\Big)R^3 + 6[\beta+(\gamma_1+\gamma_2)R]p^2 - 6\gamma_1 p^3, \\ F_R &= \frac{1}{\kappa} + 2\Big(\alpha+\frac{\beta}{3}\Big)R + 6(\gamma_1+\gamma_2)p^2 + \Big(\frac{\gamma_1}{3}+\gamma_2+3\gamma_3\Big)R^2, \\ F_A &= \beta + (\gamma_1+\gamma_2)R, \qquad F_B = \gamma_1. \end{aligned} \tag{71}$$

Then, using the calculated forms of F, F_R, F_A, F_B in (57) and (58), one obtains

$$-\frac{\Lambda-\Lambda_0}{\kappa} + 2(3\alpha+\beta)\Lambda^2 + 12(\gamma_1+3\gamma_2+9\gamma_3)\Lambda^3 + [\beta + 18(\gamma_1+\gamma_2)\Lambda]p^2 - 3\gamma_1 p^3 = 0, \tag{72}$$

and

$$3(3\gamma_1-2\gamma_2)p^2 - 2[5\beta + 6(6\gamma_1+5\gamma_2)\Lambda]p - \Big[\frac{1}{\kappa} + 4(3\alpha-\beta)\Lambda - 12(3\gamma_1+\gamma_2-9\gamma_3)\Lambda^2\Big] = 0. \tag{73}$$

If the parameters of TMG (μ^2 and Λ) are tuned according to these two algebraic relations in terms of the parameters of the cubic curvature theory, then the Type-D solutions of TMG also solve the cubic curvature theory. Solving this set of equations after setting $\gamma_1 = \gamma_2 = \gamma_3 = 0$ yields the quadratic curvature gravity results that are given below in (87) and (88).

The solutions of the quadratic curvature gravity are obtained by use of the cubic curvature results. However, it is rather instructive to rederive these solutions by using the field equations of the quadratic curvature gravity because one can arrive at the results by using the classification scheme via the scalar invariants I and J as described in Sec. II, and the masses of linearized excitations around the (anti)–de Sitter spacetime appear in the formalism in a rather curious way. First of all, let us discuss the linearized modes of TMG around (A)dS. Note that there is a single spin-2 excitation in TMG with mass-squared

$$m^2_{\rm TMG} = \mu^2 + \Lambda. \tag{74}$$

This can be seen from the linearization of (13) as follows: the right-hand side vanishes for any Einstein space and the left-hand side yields

$$(\bar{\Box} - \mu^2 - 3\Lambda)S^L_{\mu\nu} = 0, \tag{75}$$

where $\bar{\Box}$ refers to the background d'Alembertian with a metric $\bar{g}_{\mu\nu}$. Keeping in mind that in three-dimensional AdS spacetime, a massless spin-2 field satisfies

$$(\bar{\Box} - 2\Lambda)h^{\rm TT}_{\mu\nu} = 0, \tag{76}$$

where $h^{\rm TT}_{\mu\nu}$ is the transverse-traceless part of the tensorial excitation $h_{\mu\nu}$. Comparing (75) and (76), the mass in (74) follows. Even though this heuristic procedure led to the correct mass, one should always check such a computation with the help of a thorough canonical procedure, since $S^L_{\mu\nu}$ is not a fundamental excitation in this theory. Canonical analysis of this theory was carried out in [32,33] which agrees with our heuristic derivation.

Now, turning to the discussion on the solutions of the quadratic curvature gravity. For spacetimes of constant scalar curvature, the trace and the traceless field equations of the quadratic curvature gravity take the form

$$-\frac{R-6\Lambda_0}{\kappa} + \frac{3\alpha+\beta}{3}R^2 + \beta S_{\mu\nu}S^{\mu\nu} + 2K = 0, \tag{77}$$

and

$$\left(\beta\Box + \frac{1}{\kappa} + \frac{6\alpha+\beta}{3}R\right)S_{\mu\nu} = 4\beta\left(S_{\mu\rho}S^{\rho}_{\nu} - \frac{1}{3}g_{\mu\nu}S_{\sigma\rho}S^{\sigma\rho}\right), \tag{78}$$

by using (B3) and (B4). Then, if we require that the solution of the quadratic curvature gravity is also a solution of the cosmological TMG, one gets a quadratic constraint

$$S_{\mu\rho}S^{\rho}_{\nu} - \frac{1}{3}g_{\mu\nu}S_{\sigma\rho}S^{\sigma\rho} = \left(\frac{1}{\kappa\beta} + \mu^2 + 5\Lambda + \frac{12\Lambda\alpha}{\beta}\right)S_{\mu\nu}, \tag{79}$$

by using the field equations of TMG which are $R = 6\Lambda$ and (13) in (78). Besides, one can also use (77) in order to further reduce (79) to

$$\beta S_{\mu\rho}S^{\rho}_{\nu} - \left(\frac{1}{\kappa} + (\mu^2 + 5\Lambda)\beta + 12\Lambda\alpha\right)S_{\mu\nu} + \left(\frac{2}{\kappa}(\Lambda_0 - \Lambda) + 4(3\alpha+\beta)\Lambda^2\right)g_{\mu\nu} = 0. \tag{80}$$

This will serve as the main equation in classifying the solutions of quadratic gravity that are also solutions of TMG. The scalar invariants I and J can be read from (80) as

$$\begin{aligned} I &= -\frac{3}{\beta}\left(\frac{2}{\kappa}(\Lambda_0 - \Lambda) + 4(3\alpha+\beta)\Lambda^2\right), \\ J &= \left(\frac{1}{\kappa\beta} + \mu^2 + 5\Lambda + \frac{12\Lambda\alpha}{\beta}\right)I. \end{aligned} \tag{81}$$

With these we can rewrite (80) as

$$S_{\mu\rho}S^{\rho}_{\nu} - \frac{J}{I}S_{\mu\nu} - \frac{I}{3}g_{\mu\nu} = 0. \tag{82}$$

It is interesting to note that $\frac{J}{I}$ is exactly the square of the mass difference of the spin-2 excitations of TMG and quadratic gravity theories,[11] namely,

$$J = (m^2_{\rm TMG} - m^2_{\rm quadratic})I, \tag{83}$$

where $m^2_{\rm TMG} = \mu^2 + \Lambda$ [32,33] and $m^2_{\rm quadratic} = -\frac{1}{\kappa\beta} - 4\Lambda - \frac{12\Lambda\alpha}{\beta}$ [35].

What is achieved up to this point is that if a given $S_{\mu\nu}$ satisfies the TMG equations and the quadratic constraint (80), then it also satisfies the general quadratic gravity equations. Therefore, we can use the solutions of TMG which were compiled and classified in [23]. As noted in Sec. II, the Type-N spacetimes satisfy $I = J = 0$. Then, from (80), one obtains

$$\mu^2 = -\frac{1}{\kappa\beta} - \Lambda\left(5 + \frac{12\alpha}{\beta}\right), \tag{84}$$

where the effective cosmological constant reads

$$\Lambda = \frac{1}{4(3\alpha+\beta)\kappa}\left(1 \pm \sqrt{1 - 8(3\alpha+\beta)\kappa\Lambda_0}\right). \tag{85}$$

These equations relate the parameters of Type-N solutions of TMG and the Type-N solutions of the general quadratic theory. In terms of the massive spin-2 excitations of the theories, Type-N solutions satisfy the interesting property $m^2_{\rm TMG} = m^2_{\rm quadratic}$. When one sets $8\alpha + 3\beta = 0$, one gets the NMG result given in [9–11] after identifying $m^2 \equiv -\frac{1}{\kappa\beta}$.

On the other hand, for the Type-D spacetimes, by using (8) and (55) in (82), we have two equations valid for both Type-D cases,

[11] There is also a spin-0 excitation of general quadratic gravity theory with mass $m_s^2 = \frac{1}{\kappa(8\alpha+3\beta)} - 4\Lambda\left(\frac{3\alpha+\beta}{8\alpha+3\beta}\right)$ which decouples in the NMG limit [34,35].

$$\left(p^2 - \frac{J}{I}p - \frac{I}{3}\right)g_{\mu\nu} = 0, \qquad \left(p + \frac{J}{I}\right)\xi_\mu\xi_\nu = 0, \quad (86)$$

from which it follows that $I^3 = 6J^2$. These relations yield

$$\mu^2 = -\frac{9}{10\kappa\beta} - \frac{27\Lambda}{5}\left(1 + \frac{2\alpha}{\beta}\right), \quad (87)$$

where the effective cosmological constant reads

$$\Lambda = \frac{1}{12(2\alpha+\beta)(\alpha+3\beta)\kappa}\Big(-2\alpha + 9\beta \pm 5\sqrt{\beta(-2\alpha+3\beta-8(2\alpha+\beta)(\alpha+3\beta)\kappa\Lambda_0)}\Big). \quad (88)$$

In the NMG limit, these relations yield the corresponding equations in [10,12]. Let us summarize our results for the quadratic curvature gravity as Type-N and Type-D solutions of the TMG also solve the general quadratic gravity if the TMG parameter and the cosmological constant are tuned as (84) and (85) and (87) and (88), respectively.

B. Solutions of BINMG

The action of the Born-Infeld extension of NMG (BINMG) is [4]

$$I_{\mathrm{BINMG}} = -\frac{4\tilde{m}^2}{\kappa^2}\int d^3x\left[\sqrt{-\det\left(g - \frac{1}{\tilde{m}^2}G\right)} - \left(1 - \frac{\tilde{\lambda}}{2}\right)\sqrt{-\det g}\right], \quad (89)$$

where G is the Einstein tensor with components $G_{\mu\nu} \equiv R_{\mu\nu} - \frac{1}{2}g_{\mu\nu}R$. Note that we have used the tilded versions of the parameters to avoid possible confusion with the NMG parameters. By using the exact expansion

$$\det A = \frac{1}{6}[(\mathrm{Tr}A)^3 - 3\,\mathrm{Tr}A\,\mathrm{Tr}(A^2) + 2\,\mathrm{Tr}(A^3)], \quad (90)$$

which is valid for 3×3 matrices, (89) can be rewritten as [36]

$$I_{\mathrm{BINMG}} = -\frac{4\tilde{m}^2}{\kappa^2}\int d^3x\sqrt{-\det g}\,F(R, A, B), \quad (91)$$

where

$$F(R,A,B) \equiv \sqrt{\left(1 + \frac{R}{6\tilde{m}^2}\right)^3 - \frac{A}{2\tilde{m}^4}\left(1 + \frac{R}{6\tilde{m}^2}\right) - \frac{B}{3\tilde{m}^6}} - \left(1 - \frac{\tilde{\lambda}}{2}\right). \quad (92)$$

Now, let us find the constant curvature Type-N and Type-D solutions in this theory by using the formalism developed above for the generic $f(R^\mu_\nu)$ theory.

1. Type-N solutions

To begin with, for Type-N spacetimes, the functions F, F_R, F_A take the forms

$$F = \left(1 + \frac{\Lambda}{\tilde{m}^2}\right)^{(3/2)} - \left(1 - \frac{\tilde{\lambda}}{2}\right),$$
$$F_R = \frac{1}{4\tilde{m}^2}\left(1 + \frac{\Lambda}{\tilde{m}^2}\right)^{(1/2)}, \quad (93)$$
$$F_A = -\frac{1}{4\tilde{m}^4}\left(1 + \frac{\Lambda}{\tilde{m}^2}\right)^{-(1/2)},$$

after setting $R = 6\Lambda$ and with the requirement $\Lambda > -\tilde{m}^2$ which will be the lower bound on the scalar curvature. Before analyzing the solutions, let us note an observation. If one uses (93) in (44), the traceless field equation of BINMG for Type-N spacetimes of constant curvature becomes

$$(\Box - \tilde{m}^2 - 3\Lambda)S_{\mu\nu} = 0, \quad (94)$$

which is the traceless field equation of TMG in the quadratic form for the same type of spacetimes given in (14) with $\tilde{m}^2 = \mu^2$. Actually, without any calculation, one can see that the Type-N solution found in [11], where $\xi_\nu = \partial_\nu$ is not a null-Killing-vector field, is a solution of BINMG, since the traceless field equations are equivalent and the trace field equation just determines the value of the scalar curvature. More explicitly, one can use the results of Sec. IV A: inserting (93) in (45) and (46), then solving the resulting equations yields

$$\Lambda = -\tilde{m}^2\tilde{\lambda}\left(1 - \frac{\tilde{\lambda}}{4}\right), \qquad \tilde{\lambda} < 2, \quad (95)$$

and

$$\mu^2 = \tilde{m}^2, \quad (96)$$

which is the expected result in the light of the observation above. Let us give the Type-N solution of BINMG, inherited from NMG [11], corresponding to negative constant curvature as $\Lambda = -\nu^2$:

$$ds^2 = d\rho^2 + \frac{2}{\nu^2 - \beta^2}du dv + \left(Z(u,\rho) - \frac{v^2}{\nu^2 - \beta^2}\right)du^2, \quad (97)$$

where β can be either $\beta = \nu\tanh(\nu\rho)$ or $\beta = \nu\coth(\nu\rho)$, and $Z(u,\rho)$ is

$$Z(u,\rho) = \frac{1}{\sqrt{\nu^2 - \beta^2}}(\cosh(\tilde{m}\rho)F_1(u) + \sinh(\tilde{m}\rho)F_2(u) + \cosh(\nu\rho)f_1(u) + \sinh(\nu\rho)f_2(u)). \quad (98)$$

Note that, for BINMG, the solutions corresponding to the special limits $\tilde{m}^2 = -\Lambda$ and $\tilde{m}^2 = 0$ are not allowed. As described in [11], the metric

$$ds^2 = d\rho^2 + 2e^{2\nu\rho}dudv + (e^{\nu\rho}\cosh(\tilde{m}\rho)F_1(u) + e^{\nu\rho}\sinh(\tilde{m}\rho)F_2(u) + e^{2\nu\rho}f_1(u) + f_2(u))du^2 \tag{99}$$

can be obtained from (97) by taking a specific limit for which ∂_v is a Killing vector. This metric represents the AdS pp-wave solution given in [37]. As the $\tilde{m}^2 = -\Lambda$ limit is not possible for BINMG, the corresponding logarithmic solutions (in the Poincaré coordinates) for the AdS pp-wave solution are not available as discussed in [37].

2. Type-D solutions

First, one needs to calculate the functions F, F_R, F_A, F_B for Type-D spacetimes which take the forms

$$\begin{aligned}
F &= \sqrt{\left(1+\frac{R}{6\tilde{m}^2}+\frac{2}{\tilde{m}^2}p\right)\left(1+\frac{R}{6\tilde{m}^2}-\frac{p}{\tilde{m}^2}\right)^2} - \left(1-\frac{\tilde{\lambda}}{2}\right),\\
F_R &= \frac{1}{4\tilde{m}^2}\left(F+1-\frac{\tilde{\lambda}}{2}\right)^{-1}\left[\left(1+\frac{R}{6\tilde{m}^2}\right)^2 - \frac{p^2}{\tilde{m}^4}\right],\\
F_A &= -\frac{1}{4\tilde{m}^4}\left(F+1-\frac{\tilde{\lambda}}{2}\right)^{-1}\left(1+\frac{R}{6\tilde{m}^2}\right),\\
F_B &= -\frac{1}{6\tilde{m}^6}\left(F+1-\frac{\tilde{\lambda}}{2}\right)^{-1},
\end{aligned} \tag{100}$$

with the requirements $R \neq 6(p-\tilde{m}^2)$ and $R > -6(\tilde{m}^2+2p)$ which will provide a bound on the scalar curvature. For Type-D spacetimes of constant scalar curvature, the traceless field equation of BINMG can be found by using (100) in (52) as

$$\left(\Box - \tilde{m}^2 - \frac{R}{2} + 3p\right)S_{\mu\nu} = 0, \tag{101}$$

which is the traceless field equation of TMG in the quadratic form for the same type of spacetimes given in (15) with $\tilde{m}^2 = \mu^2$. Again, without any calculation, one can see that Type-D solutions of TMG are also solutions of BINMG with the condition (55) which now reads as

$$p = \frac{\tilde{m}^2}{9} + \frac{R}{6}, \tag{102}$$

but with a constant scalar curvature that is a solution of the trace field equation of BINMG. Putting this observation aside, one can find the Type-D solutions of BINMG by directly using the results obtained in Sec. IV B. In order to have Type-D solutions of BINMG which are also Type-D solutions of TMG, (57) and (58) are the equations that need to be satisfied. Using (100), (57) and (58) reduce to

$$\left(F+1-\frac{\tilde{\lambda}}{2}\right)^{-1}\left[\left(1+\frac{R}{6\tilde{m}^2}\right)^2 - \frac{p^2}{\tilde{m}^4}\right] - \left(1-\frac{\tilde{\lambda}}{2}\right) = 0, \tag{103}$$

$$\left(F+1-\frac{\tilde{\lambda}}{2}\right)^{-1}\left(1+\frac{R}{6\tilde{m}^2}-\frac{p}{\tilde{m}^2}\right)\left(1+\frac{3}{2\tilde{m}^2}R-\frac{9}{\tilde{m}^2}p\right) = 0. \tag{104}$$

The solution of the second equation is equivalent to (102) with the requirement that the scalar curvature is bounded as $R > -\frac{22}{9}\tilde{m}^2$. Putting this result in the first equation yields the constant scalar curvature as

$$R = \frac{9}{16}\tilde{m}^2\left[\left(\tilde{\lambda}^2 - 4\tilde{\lambda} - \frac{52}{27}\right) \pm (\tilde{\lambda}-2)\sqrt{\left(\tilde{\lambda}-\frac{2}{9}\right)\left(\tilde{\lambda}-\frac{34}{9}\right)}\right], \tag{105}$$

where $\tilde{\lambda} < 2/9$.[12] The Type-D solutions of TMG is parametrized with μ and R. Hence, we need to write μ in terms of the parameters of BINMG which can be achieved by using (102) in (55), and one gets

$$\mu^2 = \tilde{m}^2. \tag{106}$$

By using the Type-D solutions of TMG given in [23], the Type-D solution of BINMG with a timelike Killing vector and a negative constant scalar curvature $R \equiv -6\nu^2$ can be given as

$$ds^2 = -\left(dt + \frac{6\tilde{m}}{\tilde{m}^2+27\nu^2}\cosh\theta d\phi\right)^2 + \frac{9}{\tilde{m}^2+27\nu^2}(d\theta^2 + \sinh^2\theta d\phi^2), \tag{107}$$

while the Type-D solution of BINMG with a spacelike Killing vector and a negative constant scalar curvature reads

$$ds^2 = \frac{9}{\tilde{m}^2+27\nu^2}(-\cosh^2\rho d\tau^2 + d\rho^2) + \left(dy + \frac{6\tilde{m}}{\tilde{m}^2+27\nu^2}\sinh\rho d\tau\right)^2. \tag{108}$$

Now, let us discuss the Type-D solutions of BINMG which are also Type-D solutions of NMG but not TMG. In order to have such solutions, the set of equations that need to be satisfied is either (63) or (64). In the case of the NMG solution corresponding to the set (63), $p = -\frac{R}{3}$ should be satisfied. Using this condition and (100), the set (63) reduces to

$$\tilde{\lambda} = 2 - 2\sqrt{\left(1-\frac{R}{2\tilde{m}^2}\right)\left(1+\frac{R}{2\tilde{m}^2}\right)^2}, \tag{109}$$

$$\left(F+1-\frac{\tilde{\lambda}}{2}\right)^{-1}\left(1+\frac{R}{2\tilde{m}^2}\right)\left(1-\frac{3}{2\tilde{m}^2}R\right) = 0, \tag{110}$$

[12]The interval $\tilde{\lambda} > 34/9$ is not valid, since employing (102) in (103) yields $\sqrt{11/9 + R/(2\tilde{m}^2)} = (1-\tilde{\lambda}/2)^{-1}[10/9 + R/(3\tilde{m}^2)]$ which implies $\tilde{\lambda} < 2$ together with $R > -22\tilde{m}^2/9$.

which has the solution

$$R = \frac{2}{3}\tilde{m}^2, \qquad \tilde{\lambda} = 2 - \frac{8\sqrt{6}}{9}. \tag{111}$$

The solutions given in [12] are parametrized with m which is related to $\tilde{m}$ as $m^2 = \frac{5}{6}\tilde{m}^2$. Then, with the solutions in [12], the following two metrics are the solution of BINMG:

$$ds^2 = -d\tau^2 + e^{(2/\sqrt{3})\tilde{m}\tau}dx^2 + e^{-(2/\sqrt{3})\tilde{m}\tau}dy^2, \tag{112}$$

$$ds^2 = \cos((2/\sqrt{3})\tilde{m}x)(-dt^2 + dy^2) + dx^2 + 2\sin((2/\sqrt{3})\tilde{m}x)dtdy. \tag{113}$$

On the other hand, in the case of the NMG solution corresponding to the set (64), the relation between p and R that should be satisfied is $p = \frac{R}{6}$ for which (64) reduces to

$$\tilde{\lambda} = 2 - 2\sqrt{1 + \frac{R}{2\tilde{m}^2}}, \qquad \left(1 + \frac{R}{2\tilde{m}^2}\right)^{-(1/2)} = 0, \tag{114}$$

where the second equation does not have a solution. Therefore, just like TMG [38], BINMG does not have a constant scalar curvature Type-D solution with a hypersurface orthogonal Killing vector.

VI. CONCLUSIONS

We have shown that constant scalar curvature Type-N and Type-D solutions of topologically massive gravity and new massive gravity solve also the equations of the generic higher curvature gravity built on the contractions of the Ricci tensor in $2+1$ dimensions. Our construction is based on inheriting the previously studied solutions of the topologically massive gravity and the new massive gravity. The crux of the argument presented here is to reduce the highly complicated higher derivative equations of the $f(R_{\mu\nu})$ theory to a nonlinear wavelike equation in the traceless-Ricci tensor accompanied with a constant trace equation, and to implement the defining conditions of the Type-N and Type-D spacetimes along with the condition of the constancy of the scalar curvature. Save for the actions which include the contractions of the derivatives of the Ricci tensor, all the three-dimensional gravity theories that are based on the Ricci tensor are covered in this work. As explicit examples, we have given the solutions of the Born-Infeld extensions of the new massive gravity. Note that with our approach one can also find solutions of the generic $f(R_{\mu\nu})$ theory that fall into the other types such as Type III and Type I under the condition that the scalar curvature is constant. In this work, we have focused on the Type-N and Type-D solutions of the $f(R_{\mu\nu})$ theory, since the corresponding solutions of TMG and NMG are well studied. But, nonconstant scalar curvature solutions can be found by using the techniques developed in [9].

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APPENDIX A: SOME RELEVANT VARIATIONS

Variations of the three cubic curvature terms are

$$\delta(R^\mu_\nu R^\rho_\mu R^\nu_\rho) = 3\bigg[R^\rho_\mu R_{\rho\alpha}R^\alpha_\nu + \frac{1}{2}\Big(g_{\mu\nu}R^{\beta\rho}R^\alpha_\rho\nabla_\beta\nabla_\alpha + R^\rho_\nu R_{\mu\rho}\Box - 2R^\rho_\nu R^\alpha_\rho\nabla_\mu\nabla_\alpha\Big)\bigg]\delta g^{\mu\nu}, \tag{A1}$$

$$\delta(RR^\mu_\nu R^\nu_\mu) = R[(g_{\mu\nu}R^{\alpha\beta}\nabla_\beta\nabla_\alpha + R_{\mu\nu}\Box - 2R^\alpha_\nu\nabla_\mu\nabla_\alpha) + 2R^\rho_\nu R_{\mu\rho}]\delta g^{\mu\nu} + R^\alpha_\beta R^\beta_\alpha[(g_{\mu\nu}\Box - \nabla_\mu\nabla_\nu) + R_{\mu\nu}]\delta g^{\mu\nu}, \tag{A2}$$

$$\delta(R^3) = 3R^2[(g_{\mu\nu}\Box - \nabla_\mu\nabla_\nu) + R_{\mu\nu}]\delta g^{\mu\nu}. \tag{A3}$$

One can calculate $\delta S_{\alpha\beta}$ by using

$$\delta R_{\alpha\beta} = \frac{1}{2}(g_{\mu\nu}\nabla_\alpha\nabla_\beta + g_{\mu\alpha}g_{\beta\nu}\Box - g_{\beta\nu}\nabla_\mu\nabla_\alpha - g_{\alpha\nu}\nabla_\mu\nabla_\beta)\delta g^{\mu\nu}, \tag{A4}$$

$$\delta R = [R_{\mu\nu} + (g_{\mu\nu}\Box - \nabla_\mu\nabla_\nu)]\delta g^{\mu\nu}, \tag{A5}$$

as

$$\delta S_{\alpha\beta} = \frac{1}{2}\bigg[(g_{\mu\nu}\nabla_\alpha\nabla_\beta + g_{\mu\alpha}g_{\beta\nu}\Box - g_{\beta\nu}\nabla_\mu\nabla_\alpha - g_{\alpha\nu}\nabla_\mu\nabla_\beta) - \frac{2}{3}g_{\alpha\beta}(g_{\mu\nu}\Box - \nabla_\mu\nabla_\nu)\bigg]\delta g^{\mu\nu} + \frac{1}{3}\bigg[\Big(g_{\mu\alpha}g_{\nu\beta} - \frac{1}{3}g_{\alpha\beta}g_{\mu\nu}\Big)R - g_{\alpha\beta}S_{\mu\nu}\bigg]\delta g^{\mu\nu}. \tag{A6}$$

With this result, $\delta A \equiv \delta(S^\alpha_\beta S^\beta_\alpha)$ and $\delta B \equiv \delta(S^\alpha_\rho S^\beta_\alpha S^\rho_\beta)$ become

$$\delta A = \bigg[2\Big(S^\alpha_\mu S_{\alpha\nu} + \frac{1}{3}RS_{\mu\nu}\Big) + (g_{\mu\nu}S^{\alpha\beta}\nabla_\alpha\nabla_\beta + S_{\mu\nu}\Box - 2S^\alpha_\nu\nabla_\mu\nabla_\alpha)\bigg]\delta g^{\mu\nu}, \tag{A7}$$

$$\delta B = \left[\frac{3}{2}(g_{\mu\nu}S^{\alpha}_{\rho}S^{\beta\rho}\nabla_\alpha\nabla_\beta + S_{\mu\rho}S^{\rho}_{\nu}\Box - 2S^{\alpha}_{\rho}S^{\rho}_{\nu}\nabla_\mu\nabla_\alpha) - S^{\alpha}_{\rho}S^{\rho}_{\alpha}(g_{\mu\nu}\Box - \nabla_\mu\nabla_\nu)\right]\delta g^{\mu\nu} + \left[3S^{\rho}_{\mu}S^{\sigma}_{\rho}S_{\nu\sigma} - S^{\alpha}_{\rho}S^{\rho}_{\alpha}S_{\mu\nu} + \left(S_{\mu\rho}S^{\rho}_{\nu} - \frac{1}{3}g_{\mu\nu}S^{\alpha}_{\rho}S^{\rho}_{\alpha}\right)R\right]\delta g^{\mu\nu}. \tag{A8}$$

APPENDIX B: FIELD EQUATIONS OF CUBIC CURVATURE GRAVITY

The action (65) yields the source-free field equations with the help of the variations above:

$$\frac{1}{\kappa}\left(R_{\mu\nu} - \frac{1}{2}g_{\mu\nu}R + \Lambda_0 g_{\mu\nu}\right) + 2\alpha R\left(R_{\mu\nu} - \frac{1}{4}g_{\mu\nu}R\right) + (2\alpha + \beta)(g_{\mu\nu}\Box - \nabla_\mu\nabla_\nu)R + \beta\Box\left(R_{\mu\nu} - \frac{1}{2}g_{\mu\nu}R\right) + 2\beta\left(R_{\mu\sigma\nu\rho} - \frac{1}{4}g_{\mu\nu}R_{\sigma\rho}\right)R^{\sigma\rho} + K_{\mu\nu} = 0, \tag{B1}$$

where $\Box = \nabla_\alpha\nabla^\alpha$. The field equation for the quadratic curvature part is given in [39] and the contribution from the cubic curvature part reads

$$\begin{aligned} K_{\mu\nu} = {}& \gamma_1\left[\frac{3}{2}g_{\mu\nu}\nabla_\alpha\nabla_\beta(R^{\beta\rho}R^{\alpha}_{\rho}) + \frac{3}{2}\Box(R^{\rho}_{\nu}R_{\mu\rho}) - 3\nabla_\alpha\nabla_{(\mu}(R^{\rho}_{\nu)}R^{\alpha}_{\rho}) + 3R^{\rho}_{\mu}R_{\rho\alpha}R^{\alpha}_{\nu} - \frac{1}{2}g_{\mu\nu}R^{\alpha}_{\beta}R^{\rho}_{\alpha}R^{\beta}_{\rho}\right] \\ &+ \gamma_2\left[g_{\mu\nu}\nabla_\alpha\nabla_\beta(RR^{\alpha\beta}) + \Box(RR_{\mu\nu}) - 2\nabla_\alpha\nabla_{(\mu}(R^{\alpha}_{\nu)}R) + (g_{\mu\nu}\Box - \nabla_\mu\nabla_\nu)R^2_{\alpha\beta} + 2RR^{\rho}_{\nu}R_{\mu\rho} + R_{\mu\nu}R^{\alpha}_{\beta}R^{\beta}_{\alpha} - \frac{1}{2}g_{\mu\nu}RR^{\alpha}_{\beta}R^{\beta}_{\alpha}\right] \\ &+ \gamma_3\left[3(g_{\mu\nu}\Box - \nabla_\mu\nabla_\nu)R^2 + 3R^2R_{\mu\nu} - \frac{1}{2}g_{\mu\nu}R^3\right]. \end{aligned} \tag{B2}$$

It is quite useful to recast them into a pure trace and a traceless part as

$$(8\alpha + 3\beta)\Box R - \frac{R - 6\Lambda_0}{\kappa} + \frac{3\alpha + \beta}{3}R^2 + \beta S_{\mu\nu}S^{\mu\nu} + 2K = 0, \tag{B3}$$

and

$$\left(\beta\Box + \frac{1}{\kappa} + \frac{6\alpha + \beta}{3}R\right)S_{\mu\nu} = 4\beta\left(S_{\mu\rho}S^{\rho}_{\nu} - \frac{1}{3}g_{\mu\nu}S_{\sigma\rho}S^{\sigma\rho}\right) + (2\alpha + \beta)\left(\nabla_\mu\nabla_\nu - \frac{1}{3}g_{\mu\nu}\Box\right)R - \left(K_{\mu\nu} - \frac{1}{3}g_{\mu\nu}K\right), \tag{B4}$$

where $K \equiv g^{\mu\nu}K_{\mu\nu}$. In deriving the quadratic curvature contribution to these equations which has a Riemann tensor in it (B1), one makes use of the relation between the Riemann tensor, the traceless-Ricci tensor, and the scalar curvature in three dimensions:

$$R_{\mu\nu\rho\sigma} = g_{\mu\rho}S_{\nu\sigma} + g_{\nu\sigma}S_{\mu\rho} - g_{\nu\rho}S_{\mu\sigma} - g_{\mu\sigma}S_{\nu\rho} + \frac{1}{6}(g_{\mu\rho}g_{\nu\sigma} - g_{\mu\sigma}g_{\nu\rho})R. \tag{B5}$$

The trace part, K, and the traceless part of $K_{\mu\nu}$ are given in terms of the traceless-Ricci tensor as

$$K = \left(\frac{3}{2}\gamma_1 + 2\gamma_2\right)\Box(S^{\mu}_{\nu}S^{\nu}_{\mu}) + \frac{3}{2}\gamma_1(\nabla_\alpha\nabla_\beta + S_{\alpha\beta})(S^{\alpha\rho}S^{\beta}_{\rho}) + (\gamma_1 + \gamma_2)\left(\nabla_\alpha\nabla_\beta + \frac{3}{2}S_{\alpha\beta}\right)(RS^{\alpha\beta}) + 2\left(\frac{\gamma_1}{3} + \gamma_2 + 3\gamma_3\right)\left(\Box + \frac{R}{4}\right)R^2, \tag{B6}$$

$$\begin{aligned} K_{\mu\nu} - \frac{1}{3}g_{\mu\nu}K = {}& \left(\Box + \frac{2}{3}R\right)\left[\frac{3}{2}\gamma_1\left(S_{\mu\rho}S^{\rho}_{\nu} - \frac{1}{3}g_{\mu\nu}S^{\rho}_{\sigma}S^{\sigma}_{\rho}\right) + (\gamma_1 + \gamma_2)RS_{\mu\nu}\right] \\ &- \nabla_\alpha\left(g_{\beta(\nu}\nabla_{\mu)} - \frac{1}{3}g_{\mu\nu}\nabla_\beta\right)[3\gamma_1 S^{\alpha}_{\rho}S^{\rho\beta} + 2(\gamma_1 + \gamma_2)RS^{\alpha\beta}] \\ &+ \left(\frac{1}{3}g_{\mu\nu}\Box - \nabla_\mu\nabla_\nu + S_{\mu\nu}\right)\left[\gamma_2 S^{\rho}_{\sigma}S^{\sigma}_{\rho} + \left(\frac{\gamma_1}{3} + \gamma_2 + 3\gamma_3\right)R^2\right] \\ &+ 3\gamma_1 S^{\rho}_{\sigma}\left(S_{\mu\rho}S^{\sigma}_{\nu} - \frac{1}{3}g_{\mu\nu}S^{\alpha}_{\rho}S^{\sigma}_{\alpha}\right) + 2(\gamma_1 + \gamma_2)R\left(S_{\mu\rho}S^{\rho}_{\nu} - \frac{1}{3}g_{\mu\nu}S^{\rho}_{\sigma}S^{\sigma}_{\rho}\right). \end{aligned} \tag{B7}$$

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